1 Stieltjes transform

From last time, we see that to prove the semicircle law, it suffices to show that for all z in the upper half-plane, $s_n(z) \to s_{\mu_{\mathrm{sc}}}(z)$ almost surely. By directly controlling $s_n(z)$, the Stieltjes transform method can be used to find the semicircle law, even if we do not know this law in advance.

Let $z = a + bi$, $b > 0$. The main idea is to compare $s_n(z)$ to $s_{n-1}(z)$. Recall the Cauchy interlacing law, which says for any $n \times n$ Hermitian matrix with $(n-1) \times (n-1)$ minor A_{n-1}, one has $\lambda_i(A_n) \le \lambda_i(A_{n-1}) \le \lambda_{i+1}(A_n)$ for all $i = 1, \ldots, n-1$. The following "alternating" sum

$$\sum_{j=1}^{n-1} \frac{b}{(\lambda_j(M_{n-1})/\sqrt{n} - a)^2 + b^2} - \sum_{j=1}^{n} \frac{b}{(\lambda_j(M_n)/\sqrt{n} - a)^2 + b^2}$$

is bounded in n, because the function $x \mapsto \frac{b}{(x-a)^2+b^2}$ has finite total variation, and

$$\{\lambda_1(M_n)/\sqrt{n}, \lambda_1(M_{n-1})/\sqrt{n}, \ldots, \lambda_{n-1}(M_{n-1})/\sqrt{n}, \lambda_n(M_n)/\sqrt{n}\}$$

forms a partition. Note that the above sum is the imaginary part of

$$\sqrt{n(n-1)}s_{n-1}\left(\frac{\sqrt{n}}{\sqrt{n-1}}(a+bi)\right) - ns_n(a+bi).$$

To see this, note that for instance

$$\begin{aligned}\sqrt{n(n-1)}s_{n-1}\left(\frac{\sqrt{n}}{\sqrt{n-1}}(a+bi)\right) &= \sqrt{n(n-1)}\frac{1}{n-1}\sum_{j=1}^{n-1} \frac{1}{\lambda_j(\frac{1}{\sqrt{n-1}}M_{n-1}) - \frac{\sqrt{n}}{\sqrt{n-1}}(a+bi)} \\ &= \sum_{j=1}^{n} \frac{1}{\lambda_j(M_{n-1})/\sqrt{n} - (a+bi)}.\end{aligned}$$

The real part can be bounded similarly. Therefore, we see that

$$\sqrt{n(n-1)}s_{n-1}\left(\frac{\sqrt{n}}{\sqrt{n-1}}(a+bi)\right) - ns_n(a+bi) = O(1)$$

as $n \to \infty$. Also, note that the Stieltjes transform is smooth and $\frac{\sqrt{n}}{\sqrt{n-1}} - 1 = O(1/n)$, and so $s_{n-1}\left(\frac{\sqrt{n}}{\sqrt{n-1}}(a+bi)\right)$ and $s_{n-1}(a+bi)$ are close, and hence we can conclude that

$$s_n(a+bi) = s_{n-1}(a+bi) + O\left(\frac{1}{n}\right). \tag{1.1}$$

In particular, we see that s_n is stable in n. Moreover, the right hand side depends only on the top left $(n-1) \times (n-1)$ minor, and it is independent of the n-th row and n-th column of the matrix.

We would like to know what s_n converges to. Recall McDiarmid's inequality:

Theorem 1.1. *Let $X_1, \dots, X_n$ be independent random variables taking values in ranges $R_1, \dots, R_n$, and let $F : R_1 \times \cdots \times R_n \to \mathbb{C}$ be a function having bounded differences. That is, there exist constants $c_1, \dots, c_n$ such that for all i,*

$$|F(x_1, \dots, x_n) - F(x_1, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_n)| \leq c_i$$

for all $x_i, x_i' \in R_i$. Then for any $\lambda > 0$, one has

$$\mathbf{P}(|F(X) - \mathbf{E}F(X)| \geq \lambda\sigma) \leq C\exp(-c\lambda^2)$$

for some $C, c > 0$, where $\sigma^2 := \sum_{i=1}^n c_i^2$.

Note that the X_j's above can be vectors of different lengths. Now, observe that (1.1) still holds if we resample the n-th row and the n-th column. Write $s_n'(z)$ for the Stieltjes transform of the resampled matrix. Then by (1.1), we have $s_n(z) = s_n'(z) + O(1/n)$. Moreover, if we interchange the n-th row and j-th row, and also n-th column and j-th column, so that the matrix is still Hermitian and has the same distribution as before, the Stieltjes transform will remain the same. In other words, if we resample the j-th row (and hence the j-th column by symmetry), the change of $s_n(z)$ is at most $O(1/n)$. Therefore, applying McDiarmid's inequality with $X_j = (\xi_{j,j}, \xi_{j,j+1}, \dots, \xi_{j,n})$, we have

$$\mathbf{P}(|s_n(z) - \mathbf{E}s_n(z)| \geq c'\lambda/\sqrt{n}) \leq C\exp(-c\lambda^2).$$

Take $\lambda = n^{1/3}$ and apply Borel-Cantelli, we obtain that almost surely,

$$s_n(z) - \mathbf{E}s_n(z) \leq O(n^{-1/6})$$

for all large n. Therefore, almost surely, $s_n(z) - \mathbf{E}s_n(z) \to 0$ for all z in the upper half-plane.

Thus, it remains to study what $\mathbf{E}s_n(z)$ tends to as $n \to \infty$. Note that

$$\mathbf{E}s_n(z) = \frac{1}{n}\sum_{j=1}^n \mathbf{E}\left[\left(\frac{1}{\sqrt{n}}M_n - zI_n\right)^{-1}_{j,j}\right]$$

Again, interchanging the rows and columns so that the matrix M_n is still Wigner will not alter the distribution of the matrix. In particular, the (j,j)-th entry of $\left(\frac{1}{\sqrt{n}}M_n - zI_n\right)^{-1}$ has the same distribution as the (n,n)-th entry. Therefore,

$$\mathbf{E}s_n(z) = \frac{1}{n}\sum_{j=1}^n \mathbf{E}\left[\left(\frac{1}{\sqrt{n}}M_n - zI_n\right)^{-1}_{j,j}\right] = \mathbf{E}\left[\left(\frac{1}{\sqrt{n}}M_n - zI_n\right)^{-1}_{n,n}\right].$$

So we need only to study the last entry of $\left(\frac{1}{\sqrt{n}}M_n - zI_n\right)^{-1}$, and we will use a formula based on the Schur complement.

References

[1] Tao, Terence. Topics in random matrix theory. Graduate Studies in Mathematics, 132. American Mathematical Society, Providence, RI, 2012. x+282 pp. ISBN: 978-0-8218-7430-1

[2] Thompson, Brady. Talk on 4/6.