# 1 The semicircle law

Recall that to the show the semicircular law, it remains to show that almost surely, for all $k \geq 0$,

$$\int_{\mathbb{R}} x^k \, \mathrm{d}\mu_{\frac{1}{\sqrt{n}}M_n}(x) \to \int_{\mathbb{R}} x^k \, \mathrm{d}\mu_{\mathrm{sc}}(x).$$

In fact, it suffices to show this convergence for any lacunary sequence, but let's forget about this for a moment.

Also recall that

$$\int_{\mathbb{R}} x^k \, \mathrm{d}\mathbf{E}\mu_{\frac{1}{\sqrt{n}}M_n}(x) = \mathbf{E}\frac{1}{n}\mathrm{tr}\left[\left(\frac{1}{\sqrt{n}}M_n\right)^k\right].$$

We showed that when $k$ is even, the term in $\mathbf{E}\mathrm{tr}(M_n)^k$ which dominates is $C_{k/2}n(n-1)\cdots(n-k/2)$. For the remaining terms, if the class of cycle has $j$ edges, where $j < k/2$, then we had a bound $n^{j+1}K^{k-2j}$, where $K$ is an upper bound for $\xi_{i,j}$. Since we assumed that the entries are uniformly bounded (one of the reductions), we can ignore the $K$-term and so the bound becomes $n^{j+1}$. As there are $O_k(1)$ many classes, we obtain

$$\mathbf{E}\frac{1}{n}\mathrm{tr}\left[\left(\frac{1}{\sqrt{n}}M_n\right)^k\right] = C_{k/2} + O_k(1/n)$$

when $k$ is even. When $k$ is odd, it is even simpler. There is no non-crossing cycle of odd length, and by the same computation we will see that

$$\mathbf{E}\frac{1}{n}\mathrm{tr}\left[\left(\frac{1}{\sqrt{n}}M_n\right)^k\right] = O_k(1/n)$$

when $k$ is odd.

Suppose that $k$ is even. For simplicity of notation we write

$$m_k^{(n)} = \int_{\mathbb{R}} x^k \, \mathrm{d}\mu_{\frac{1}{\sqrt{n}}M_n}(x).$$

Let $\varepsilon > 0$ and consider

$$\begin{aligned}\mathbf{P}\left(\left|m_k^{(n)} - C_{k/2}\right| \geq \varepsilon\right) &\leq \frac{\mathbf{E}\left|m_k^{(n)} - C_{k/2}\right|^2}{\varepsilon^2} \\ &\leq \frac{2}{\varepsilon^2}\left[\mathrm{Var}\left(m_k^{(n)}\right) + \left(\mathbf{E}m_k^{(n)} - C_{k/2}\right)^2\right]\end{aligned}$$

We can apply similar argument as before to show that

$$\mathbf{E}\left[\frac{1}{n}\mathrm{tr}\left[\left(\frac{1}{\sqrt{n}}M_n\right)^k\right]\right]^2 = C_{k/2}^2 + O_k(1/n),$$

which implies

$$\mathrm{Var}\left(m_k^{(n)}\right) = \mathbf{E}\left[\frac{1}{n}\mathrm{tr}\left[\left(\frac{1}{\sqrt{n}}M_n\right)^k\right]\right]^2 - \left[\mathbf{E}\frac{1}{n}\mathrm{tr}\left[\left(\frac{1}{\sqrt{n}}M_n\right)^k\right]\right]^2 = O_k(1/n).$$

We also have

$$\left|\mathbf{E}m_k^{(n)} - C_{k/2}\right| \le O_k(1/n).$$

Thus,

$$\mathbf{P}\left(\left|m_k^{(n)} - C_{k/2}\right| \ge \varepsilon\right) \le O_k(1)\frac{1}{n}.$$

In particular, for any lacuary sequence $(n_j)$,

$$\mathbf{P}\left(\left|m_k^{(n_j)} - C_{k/2}\right| \ge \varepsilon\right) \le O_k(1)\frac{1}{n_j}.$$

Therefore, by Borel-Cantelli, almost surely, $\left|m_k^{(n_j)} - C_{k/2}\right| < \varepsilon$ for all large $j$. That is, almost surely,

$$\int_{\mathbb{R}} x^k \,\mathrm{d}\mu_{\frac{1}{\sqrt{n_j}}M_{n_j}}(x) \to C_{k/2}.$$

For $k$ odd, same argument shows

$$\int_{\mathbb{R}} x^k \,\mathrm{d}\mu_{\frac{1}{\sqrt{n_j}}M_{n_j}}(x) \to 0.$$

It remains to show

$$\int_{\mathbb{R}} x^k \,\mathrm{d}\mu_{\mathrm{sc}}(x) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ C_{k/2} & \text{if } k \text{ is even.}\end{cases}$$

The odd case is obvious. For $k$ even,

$$\begin{aligned}
\int_{\mathbb{R}} x^k \,\mathrm{d}\mu_{\mathrm{sc}}(x) &= \frac{1}{2\pi}\int_{-2}^{2} x^k\sqrt{4-x^2}\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^2 x^k\sqrt{4-x^2}\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^1 2^{k/2+1}y^{k/2}(1-y)^{1/2}y^{-1/2}\,\mathrm{d}y \quad (x = 2\sqrt{y}) \\
&= \frac{2^{k/2+1}}{\pi}\int_0^1 y^{k/2-1/2}(1-y)^{1/2}\,\mathrm{d}y \\
&= \frac{2^{k/2+1}}{\pi}\mathrm{B}(k/2+1/2, 3/2) \\
&= \frac{1}{k/2+1}\binom{k}{k/2} = C_{k/2}.
\end{aligned}$$

## 2 Stieltjes transform

We can also prove the Wigner semicircle law using Stieltjes transform. It is similar (in spirit) to using Fourier transform to prove the central limit theorem for sum of i.i.d. random variables.

For any probability measure on $\mathbb{R}$, we define the Stieltjes transform by

$$s_\mu(z) = \int_{\mathbb{R}} \frac{1}{x-z} \, \mathrm{d}\mu(x),$$

where $z \notin \operatorname{supp}(\mu)$. We will write $s_n(z) = s_{\mu_{\frac{1}{\sqrt{n}}M_n}}(z)$. Note that

$$s_n(z) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\lambda_i(M_n/\sqrt{n}) - z} = \frac{1}{n} \operatorname{tr}\left[\left(\frac{1}{\sqrt{n}} M_n - zI\right)^{-1}\right].$$

$\left(\frac{1}{\sqrt{n}} M_n - zI\right)^{-1}$ is the normalized resolvent of $M_n$, and it plays an important role in the spectral theory of $M_n$.

We first study some basic properties of Stieltjes transform.

- Obviously, $\overline{s_\mu(z)} = s_\mu(\overline{z})$. So it suffices to study $s_\mu(z)$ on the upper half-plane.
- $\operatorname{Im}(s_\mu(z)) > 0$ if $\operatorname{Im}(z) > 0$.

  *Proof.* Write $z = a + bi$. Then

  $$\operatorname{Im} \frac{1}{(x-z)} = \operatorname{Im} \frac{1}{(x-a) - bi} = \operatorname{Im} \frac{(x-a) + bi}{(x-a)^2 + b^2} > 0.$$

  □

- From above computation, one sees that

  $$\operatorname{Im}(s_\mu(a+bi)) = \pi \mu * P_b(a),$$

  where $P_b$ is the Poisson kernel

  $$P_b(x) = \frac{1}{\pi} \frac{b}{x^2 + b^2}.$$

  Since Poisson kernel is an approximation to the identity, we have $\operatorname{Im}(s_\mu(\cdot + bi)) \to \pi\mu$ as $b \to 0^+$ in weak$^*$-topology. Equivalently,

  $$\frac{s_\mu(\cdot + bi) - s_\mu(\cdot - bi)}{2\pi i} \to \mu$$

  as $b \to 0^+$ in weak$^*$-topology. Note that this doesn't require $\mu$ to be a probability measure, but just a signed measure.

- $\mu_n \Rightarrow \mu$ if and only if $s_{\mu_n}(z) \to s_\mu(z)$ for all $z$ in the upper half-plane.

  *Proof.* The only if part is easy. For the if part, write $F_n$ for the distribution function of $\mu_n$. By Helly's selection theorem, for any subsequence of $F_n$, there exists a further subsequence $F_{n_k}$ and a nondecreasing, right continuous function $F$ such that $\lim_{k\to\infty} F_{n_k}(x) = F(x)$ for all continuity points $x$ of $F$. Write $\mu_F$ for the measure with distribution function $F$. By the only if part, $s_{\mu_F}(z) = s_\mu(z)$. Then by last property,

  $$\frac{s_{\mu_F}(\cdot + bi) - s_{\mu_F}(\cdot - bi)}{2\pi i} \to \mu$$

  in weak$^*$, which implies $\mu = \mu_F$. Therefore, every subsequence of $\mu_n$ has a further subsequence that converges to $\mu$. That is, $\mu_n \Rightarrow \mu$. $\square$

# References


[1] Tao, Terence. Topics in random matrix theory. Graduate Studies in Mathematics, 132. American Mathematical Society, Providence, RI, 2012. x+282 pp. ISBN: 978-0-8218-7430-1