

1 The semicircle law

Let's recall that we want to show the semicircle law:

Theorem 1.1 (The Wigner semicircle law). *Let M_n be the top left $n \times n$ minor of an infinite Wigner matrix $(\xi_{i,j})$. Then the ESD's $\mu_{\frac{1}{\sqrt{n}}M_n}$ converges almost surely to the Wigner semicircular distribution*

$$d\mu_{\text{sc}} := \frac{1}{2\pi}(4 - |x|^2)_+^{1/2} dx.$$

Here we assume that the diagonal entries have bounded mean and variance, and the off diagonal entries have mean 0 and variance 1.

There are 3 main reductions. We can assume that

1. the coefficients are bounded;
2. the diagonal entries vanish;
3. n ranges over a lacunary sequence: it suffices to show that the convergence holds over a subsequence n_m , where $n_m := \lfloor (1 + \varepsilon)^m \rfloor$ for some $\varepsilon > 0$.

We have also proved the following lemma:

Lemma 1.2. *For all $n \times n$ Hermitian matrices A, B , for all $\lambda \in \mathbb{R}$, and for all $\varepsilon > 0$, we have*

$$\mu_{\frac{1}{\sqrt{n}}(A+B)}(-\infty, \lambda) \leq \mu_{\frac{1}{\sqrt{n}}A}(-\infty, \lambda + \varepsilon) + \frac{1}{\varepsilon^2 n^2} \|B\|_F^2$$

and

$$\mu_{\frac{1}{\sqrt{n}}(A+B)}(-\infty, \lambda) \geq \mu_{\frac{1}{\sqrt{n}}A}(-\infty, \lambda - \varepsilon) - \frac{1}{\varepsilon^2 n^2} \|B\|_F^2.$$

Let's see how this implies 1 and 2. We first show 2. Suppose that we have shown the semicircle law with matrices that the diagonal entries vanish. Take B_n to be the diagonal matrix with entries same as that of M_n and take $A_n = M_n - B_n$. Then by Lemma 1.2,

$$\mu_{\frac{1}{\sqrt{n}}M_n}(-\infty, \lambda) \leq \mu_{\frac{1}{\sqrt{n}}A_n}(-\infty, \lambda + \varepsilon) + \frac{1}{\varepsilon^2 n^2} \|B_n\|_F^2.$$

The right hand side converges almost surely to $\mu_{\text{sc}}(-\infty, \lambda + \varepsilon)$, because B_n is just sum of n i.i.d. random variables. So

$$\limsup_{n \rightarrow \infty} \mu_{\frac{1}{\sqrt{n}}M_n}(-\infty, \lambda) \leq \mu_{\text{sc}}(-\infty, \lambda + \varepsilon).$$

Similarly we have

$$\liminf_{n \rightarrow \infty} \mu_{\frac{1}{\sqrt{n}}M_n}(-\infty, \lambda) \geq \mu_{\text{sc}}(-\infty, \lambda - \varepsilon).$$

Let $\varepsilon \rightarrow 0$, we see that $\mu_{\frac{1}{\sqrt{n}}M_n} \rightarrow \mu_{\text{sc}}$ almost surely.

We may now assume that the diagonal vanishes and see why Lemma 1.2 again implies 1. Suppose that we proved the semicircle law for truncated matrices. Let $C > 0$ be a fixed

constant. Let $M_n^{(C)} = (\xi_{i,j} \mathbf{1}_{\{|\xi_{i,j}| \leq C\}})$. Apply Lemma 1.2 with $A = M_n^{(C)}$, $B = M_n - M_n^{(C)}$, we have that for any $\lambda \in \mathbb{R}$ and any $\varepsilon > 0$,

$$\mu_{\frac{1}{\sqrt{n}}M_n}(-\infty, \lambda) \leq \mu_{\frac{1}{\sqrt{n}}M_n^{(C)}}(-\infty, \lambda + \varepsilon) + \frac{1}{\varepsilon^2 n^2} \|M_n - M_n^{(C)}\|_F^2.$$

Letting $n \rightarrow \infty$, RHS converges almost surely to $\mu_{sc}^{(C)}(-\infty, \lambda + \varepsilon) + \mathbf{E}|\xi_{1,2}|^2 \mathbf{1}_{\{|\xi_{1,2}| > C\}}$, by the strong law of large numbers, and we have $\mu_{sc}^{(C)}$ instead of μ_{sc} because the truncated matrix might not have variance 1 in off diagonal entries (but yet they still have the same variance). Letting $C \rightarrow \infty$, we get

$$\limsup_{n \rightarrow \infty} \mu_{\frac{1}{\sqrt{n}}M_n}(-\infty, \lambda) \leq \mu_{sc}(-\infty, \lambda + \varepsilon).$$

Another inequality is similar.

Let's see why it suffices to assume 3. We will need the following:

Proposition 1.3 (Cauchy interlacing law). *For any $n \times n$ Hermitian matrix A_n with top left $(n-1) \times (n-1)$ minor A_{n-1} , we have*

$$\lambda_i(A_n) \leq \lambda_i(A_{n-1}) \leq \lambda_{i+1}(A_n)$$

for all $1 \leq i \leq n-1$.

This can be proved easily if you know the min-max principle for eigenvalues. Now let $\lambda \in \mathbb{R}$ and $n > m > 0$. We first claim that

$$\mu_{M_m}(-\infty, \lambda) \leq \frac{n}{m} \mu_{M_n}(-\infty, \lambda).$$

To see this, recall

$$\begin{aligned} \mu_{M_m}(-\infty, \lambda) &= \frac{1}{m} \#\{i : \lambda_i(M_m) < \lambda\} \\ &\leq \frac{1}{m} \#\{i : \lambda_i(M_n) < \lambda\} \\ &= \frac{n}{m} \mu_{M_n}(-\infty, \lambda). \end{aligned}$$

Similarly, one can show that

$$\frac{n}{m} \mu_{M_n}(-\infty, \lambda) - \frac{n-m}{m} \leq \mu_{M_m}(-\infty, \lambda).$$

Now suppose that we proved the semicircle law along a lacunary sequence, and let's see why this implies the semicircle law holds for the whole sequence. Let $\varepsilon > 0$ and take $n_j = \lfloor (1 + \varepsilon)^j \rfloor$. Take $n \in [n_{j-1}, n_j)$. Put $n = n_j$ and $m = n$ in the above claim, we have

$$\frac{n_j}{n} \mu_{\frac{1}{\sqrt{n_j}}M_{n_j}}(-\infty, \lambda/\sqrt{n_j}) - \frac{n_j - n}{n} \leq \mu_{\frac{1}{\sqrt{n}}M_n}(-\infty, \lambda/\sqrt{n}) \leq \frac{n_j}{n} \mu_{\frac{1}{\sqrt{n_j}}M_{n_j}}(-\infty, \lambda/\sqrt{n_j}).$$

Replace λ by $\sqrt{n}\lambda$ and let $j \rightarrow \infty$ (and hence $n \rightarrow \infty$), we get

$$\mu_{\text{sc}}(-\infty, \min\{\lambda, \lambda/\sqrt{1+\varepsilon}\}) - \varepsilon \leq \liminf_{n \rightarrow \infty} \mu_{\frac{1}{\sqrt{n}}M_n}(-\infty, \lambda)$$

and

$$\limsup_{n \rightarrow \infty} \mu_{\frac{1}{\sqrt{n}}M_n}(-\infty, \lambda) \leq (1+\varepsilon)\mu_{\text{sc}}(-\infty, \max\{\lambda, \lambda/\sqrt{1+\varepsilon}\}).$$

Let $\varepsilon \rightarrow 0$ we get the convergence of whole sequence.

We will now prove the semicircle law, using the moment methods and under the three assumptions. We will use the Carleman continuity theorem:

Theorem 1.4. *Let X, X_1, X_2, \dots be a sequence of uniformly subgaussian real random variables. Then the following are equivalent:*

1. For all $k \geq 0$, $\mathbf{E}X_n^k \rightarrow \mathbf{E}X^k$ as $n \rightarrow \infty$.
2. $X_n \Rightarrow X$ as $n \rightarrow \infty$.

Recall that we want to show that the ESD's converge almost surely to the semicircular law. Using above, we just need to show that almost surely, for all $k \geq 0$,

$$\int_{\mathbb{R}} x^k d\mu_{\frac{1}{\sqrt{n}}M_n}(x) \rightarrow \int_{\mathbb{R}} x^k d\mu_{\text{sc}}(x).$$

We also need to check the subgaussian hypothesis. But what are the random variables? In this case, it is just the identity function $X_n(x) = x$, and to verify where the measure is subgaussian, we just need to show that almost surely, there exists $C, c > 0$ such that

$$\mu_{\frac{1}{\sqrt{n}}M_n}(\{x \in \mathbb{R} : |x| \geq \lambda\}) \leq Ce^{-c\lambda^2}$$

for all large λ and for all n . But this is clear, since by Bai-Yin (although we did not prove), under the assumption that the entries are bounded, for any $\varepsilon > 0$, almost surely, $\|M\| \leq (2+\varepsilon)\sqrt{n}$ for all large n .

Now, observe that

$$\int_{\mathbb{R}} x^k d\mu_{\frac{1}{\sqrt{n}}M_n}(x) = \frac{1}{n} \sum_{i=1}^n \lambda_i^k \left(\frac{1}{\sqrt{n}}M_n \right) = \frac{1}{n} \text{tr} \left[\left(\frac{1}{\sqrt{n}}M_n \right)^k \right].$$

Taking expectation, we have

$$\int_{\mathbb{R}} x^k d\mathbf{E}\mu_{\frac{1}{\sqrt{n}}M_n}(x) = \mathbf{E} \frac{1}{n} \text{tr} \left[\left(\frac{1}{\sqrt{n}}M_n \right)^k \right].$$

This is exactly the moment we saw when we were finding the operator norm of random matrices. We will show that convergence of moments next time, and this will finish the proof.

References

- [1] Tao, Terence. Topics in random matrix theory. Graduate Studies in Mathematics, 132. American Mathematical Society, Providence, RI, 2012. x+282 pp. ISBN: 978-0-8218-7430-1