1 Non-crossing cycles and Catalan number

This time we will show the following lemma:

Lemma 1.1. *The number of non-crossing cycles of length k in $\{1, \dots, n\}$ is equal to $C_{k/2}n(n-1)\cdots(n-k/2)$, where $C_{k/2}$ is the number of Dyck words of length k.*

We will use a slightly different formulation, and will not follow [1] completely. How to think of non-crossing cycles? We observe that a cycle of length k is non-crossing if and only if there exists a tree in the complete graph of order n with $k/2$ edges and $k/2+1$ vertices, such that the cycle lies in the tree and traverses each edge in the tree exactly twice. It will be clearer if I draw a picture, but I am too lazy to do that here.

Why such a cycle is called non-crossing? It is because one can draw the cycles as follows. Let $(i_1, \dots, i_k)$ be a cycle of length k and arrange $1, \dots, k$ on the circle. Draw a line segment between a and b if $i_a = i_b$. If there is no crossing (and the number of line segment is $k/2-1$), the cycle will be non-crossing. Again it will be clear if one draws a picture.

Instead of Dyck words, we will consider Dyck paths. A Dyck path of length k (where k is even) is a diagonal lattice path from $(0,0)$ to $(k,0)$ that never goes below the x-axis. Note that the number of rooted trees with $k/2+1$ vertices is equal to the number of Dyck paths with k steps. To see this, we can construct a tree from the Dyck path as follows. If the path goes up, then we make a new branch from the vertex we are staying at, and if it goes down, then we just return to the parent of the vertex.

Since a non-crossing cycle gives rise to an ordered sequence and a Dyck path, and there is an obvious bijection between Dyck paths and Dyck words, this proves Lemma 1.1.

We have shown last time that for a Hermitian Wigner matrix M with entries having mean 0, variance 1, and bounded above by $o(\sqrt{n})$ in magnitude, then $\mathbf{E}\mathrm{tr}(M^k) = (C_{k/2} + o_k(1))n^{k/2+1}$ (I think in the seminar I said that this seems to be just the upper bound, but indeed this is a 2-sided bound). Combining with $\|M\|^k \le \mathrm{tr}(M^k) \le n\|M\|^k$, we have

$$\mathbf{E}\|M\|^k \ge (C_{k/2} + o_k(1))n^{k/2}.$$

How to find $C_{k/2}$? We can easily prove by induction that there exists a recurrence relation

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}.$$

Define the generating function of C_n:

$$f(z) = \sum_{n=0}^{\infty} C_n z^n.$$

Then

$$f(z)^2 = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} C_i C_{n-i} \right) z^n = \sum_{n=0}^{\infty} C_{n+1} z^n.$$

Thus we have the functional equation

$$z(f(z))^2 = f(z) - 1.$$

Solving for f, we have

$$f(z) = \frac{1 - \sqrt{1 - 4z}}{2z},$$

and by finding the coefficients of the Taylor series of f we see that

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

References

[1] Tao, Terence. Topics in random matrix theory. Graduate Studies in Mathematics, 132. American Mathematical Society, Providence, RI, 2012. x+282 pp. ISBN: 978-0-8218-7430-1

[2] Thompson, Brady. Talk on 2/23.