Last time we showed that

$$\mathbf{P}(\|M\| \geq \lambda) \leq \lambda^{-k}(k/2)^k n^{k/2+1} \max\{1, (K/\sqrt{n})^{k-2}\}$$

for all $\lambda > 0$. Put $\mathbf{P}(\|M\| \geq \lambda) = 1/2$ and solve for λ, we see that the median of $\|M\|$ satisfies $\mathbf{M}\|M\| = O(n^{1/k}k\sqrt{n}\max\{1, K/\sqrt{n}\})$. Optimize this over k, we see that this is minimum when $k = \log n$, which gives an upper bound $O(\sqrt{n}\log n \max\{1, K/\sqrt{n}\})$ for the median. By Talagrand's inequality, this tells us $\|M\| = O(\sqrt{n}\log n \max\{1, K/\sqrt{n}\})$ with high probability. To summarize, we have shown the following:

Proposition 0.1. *If M is a Hermitian Wigner matrix ensemble with $\mathbf{E}\xi_{i,j} = 0$, $\mathrm{Var}(\xi_{i,j}) \leq 1$, and $|\xi_{i,j}| \leq K$ for all i, j, then $\|M\| = O(\sqrt{n}\log n \max\{1, K/\sqrt{n}\})$ with high probability.*

When $K \leq \sqrt{n}$, this gives an upper bound of $O(\sqrt{n}\log n)$, which is close to $O(\sqrt{n})$ but off by a logarithm.

1 Improving the bound

Recall that when we were obtaining the bound

$$\mathbf{E}\mathrm{tr}(M^k) \leq (k/2)^k n^{k/2+1} \max\{1, K/\sqrt{n}\}^{k-2},$$

we saw that each class of cycle contributed a bound of $n^{j+1}K^{k-2j}$ to the above expression, where j is the number of edges in that cycle. Let's consider the case $K = o(\sqrt{n})$ and $\mathrm{Var}(\xi_{i,j}) = 1$. In this case, when $j < k/2$, such expression is $o(n^{j+1}n^{k/2-j}) = o(n^{k/2+1})$, and so the total contribution from such cycles is $o_k(n^{k/2+1})$.

When $j = k/2$, each edge is repeated exactly once. Because $\mathrm{Var}(\xi_{i,j}) = 1$, the term $\mathbf{E}\xi_{i_1,i_2}\cdots\xi_{i_k,i_1}$ is exactly 1. Therefore, the total contribution of these classes to $\mathbf{E}\mathrm{tr}(M^k)$ is equal to a purely combinatorial quantity, namely the number of cycles of length k on $\{1, 2, \ldots, n\}$ in which each edge is repeated exactly once. We will call such cycles non-crossing cycles of length k in $\{1, \ldots, n\}$. We would like to count the number of non-crossing cycles. Before that, let's see some examples.

Example 1. *Let $a, b, c, d \in \{1, \ldots, n\}$ be distinct. Then (a, b, c, d, c, b) and (a, b, a, c, a, d) are non-crossing cycles with length 6, since there are 3 distinct edges and 4 distinct vertices in each of these two cycles.*

We will count the number of non-crossing cycles via Dyck words. For a positive even integer k, a Dyck word of length k is a word consisting of left and right parentheses of length k, such that when one reads from left to right, there are always at least as many left parentheses as right parentheses, and the total number of left parentheses is the same as the total number of right parentheses. For example, the only Dych word of length 2 is (), the two Dyck words of length 4 are (()) and ()(), and the five Dyck words of length 6 are ()()(), (())(), ()(()), (()()), ((())).

Lemma 1.1. *The number of non-crossing cycles of length k in $\{1, \ldots, n\}$ is equal to $C_{k/2} n(n-1) \cdots (n-k/2)$, where $C_{k/2}$ is the number of Dyck words of length k.*

The number $C_{k/2}$ is also known as the $(k/2)$-th Catalan number, which equals

$$\frac{k!}{(k/2+1)!(k/2)!}.$$

Before proving the lemma, note that $n(n-1)\cdots(n-k/2) = o_k(n^{k/2+1})$. Therefore, we have

Theorem 1.2. *If M is a Hermitian Wigner matrix ensemble with $\mathbf{E}\xi_{i,j} = 0$, $\mathrm{Var}(\xi_{i,j}) = 1$, and $|\xi_{i,j}| \leq o(\sqrt{n})$ for all i, j, then for $k \in 2\mathbb{N}$, we have*

$$\mathbf{E}\mathrm{tr}(M^k) = (C_{k/2} + o_k(1))n^{k/2+1}.$$

Next time we should give a precise proof of Lemma 1.1.

References

[1] Ghosh, Koushik. Talk on 2/16.

[2] Tao, Terence. Topics in random matrix theory. Graduate Studies in Mathematics, 132. American Mathematical Society, Providence, RI, 2012. x+282 pp. ISBN: 978-0-8218-7430-1