1 The moment method

Last time we saw that by finding an upper bound for $\mathbf{E}\mathrm{tr}(M^4)$, we would obtain $\|M\| \lesssim n^{3/4}$ with high probability. Let's see what will happen if we consider $\mathbf{E}\mathrm{tr}(M^6)$.

Again,

$$\mathbf{E}\mathrm{tr}(M^6) = \sum_{1 \le i_1, \ldots, i_6 \le n} \mathbf{E}\xi_{i_1,i_2} \cdots \xi_{i_5,i_6}\xi_{i_6,i_1},$$

a sum over cycles of length 6 in $\{1, 2, \ldots, n\}$. As before, we just need to consider those cycles in which each edge occurs at least twice. We can divide the cycles into 4 classes:

1. There are 3 distinct edges, each occurring twice. This type of cycle give a contribution 1, and such cycles have at most 4 vertices, which give a total contribution $O(n^4)$.
2. There are 2 distinct edges, one of them occurring 4 times and the other occurs 2 times. The number of vertices involved is at most 3, so the total contribution is $O(K^2n^3)$.
3. There are 2 distinct edges and each of them occurs three times. This contributes $O(K^2n^3)$ again.
4. There is only 1 edge occurs 6 times. The contribution is $O(K^4n^2)$.

This implies

$$\mathbf{E}\mathrm{tr}(M^6) \le O(n^4) + O(K^2n^3) + O(K^4n^2).$$

If $K = O(\sqrt{n})$, this gives $\mathbf{E}\mathrm{tr}(M^6) \le O(n^4)$. Recall that

$$\|M\| \le \mathrm{tr}(M^6)^{1/6}.$$

Using similar argument as before we see that $\|M\| \le O(n^{2/3})$ with high probability.

1.1 k-th moment computation

Let's consider $\mathbf{E}\mathrm{tr}(M^k)$, where k is an even integer. Again,

$$\mathbf{E}\mathrm{tr}(M^k) = \sum_{1 \le i_1, \ldots, i_k \le n} \mathbf{E}\xi_{i_1,i_2} \cdots \xi_{i_k,i_1},$$

where the sum if taken over all cycles of length k in $\{1, 2, \ldots, n\}$. The only non-vanishing expectation are those for which each edge occurs at least twice. In particular, there are at most $k/2$ distinct edges, and thus at most $k/2 + 1$ vertices.

We would like to divide the cycles into various classes. For instance, let's again consider $k = 4$. In this case, we have terms like

$$\mathbf{E}\xi_{i_1,i_2}\xi_{i_2,i_3}\xi_{i_3,i_4}\xi_{i_4,i_1}.$$

Recall that we have several classes, one of which is $i_1 = i_3$, $i_2 = i_4$, but $i_1 \ne i_2$. In this case, all the edges $\{i_1, i_2\}$, $\{i_2, i_3\}$, $\{i_3, i_4\}$ and $\{i_4, i_1\}$ are the same. We will write $1 \sim 2 \sim 3 \sim 4$, because all the four terms are the same.

Let's do one more example, again with $k = 4$. Consider the case $i_1 = i_3$, but i_2, i_4 are distinct from each other and from i_1. In this case, the edge $\{i_1, i_2\}$ will be the same as $\{i_2, i_3\}$, and the edge $\{i_3, i_4\}$ will be the same as $\{i_4, i_1\}$. We can write $1 \sim 2$ and $3 \sim 4$ in this case, because the first and the second terms are the same, and the third and the fourth terms are the same. However, $2 \not\sim 3$.

In general, we divide the cycles into various classes, depending on which edges are equal to each other. More precisely, a class is an equivalence relation $\sim$ on a set of k labels, say $\{1, \ldots, k\}$, where each class has at least two elements. We associate a cycle of k edges $\{i_1, i_2\}$, ..., $\{i_k, i_1\}$ to a class when we have $\{i_j, i_{j+1}\} = \{i_{j'}, i_{j'+1}\}$ if and only if $j \sim j'$. For instance, in the first example we saw, the case $i_1 = i_3$, $i_2 = i_4$, but $i_1 \neq i_2$ is associated to the class $1 \sim 2 \sim 3 \sim 4$.

How many classes are there? Well, we need to assign up to $k/2$ labels to k edges, so a crude upper bound would be $(k/2)^k$.

Now consider a given class of cycle. it has j edges (and hence at most $j+1$ many vertices), where $1 \leq j \leq k/2$, with multiplicity $a_1, \ldots, a_j$. Note that $a_i \geq 2$ (because each edge appears at least twice so that the expectation will not vanish) and $\sum_{i=1}^{j} a_i = k$. Therefore,

$$\mathbf{E}\xi_{i_1,i_2} \cdots \xi_{i_k,i_1} \leq K^{a_1-2}K^{a_2-2} \cdots K^{a_j-2} = K^{k-2j},$$

and thus

$$\begin{aligned}\mathbf{E}\mathrm{tr}(M^k) &\leq (k/2)^k K^{k-2j} O(n^{j+1}) \\ &\leq (k/2)^k \max\{n^{k/2+1}, n^2 K^{k-2}\} \\ &= (k/2)^k n^{k/2+1} \max\{1, (K/\sqrt{n})^{k-2}\}.\end{aligned}$$

By $\|M\|^k \leq \mathrm{tr}(M^k)$ again, we have

$$\mathbf{E}\|M\|^k \leq (k/2)^k n^{k/2+1} \max\{1, (K/\sqrt{n})^{k-2}\}.$$

So by Markov's inequality, we have

$$\mathbf{P}(\|M\| \geq \lambda) \leq \lambda^{-k}(k/2)^k n^{k/2+1} \max\{1, (K/\sqrt{n})^{k-2}\}$$

for all $\lambda > 0$. We will continue next time.

References

[1] Ghosh, Koushik. Talk on 2/9.

[2] Tao, Terence. Topics in random matrix theory. Graduate Studies in Mathematics, 132. American Mathematical Society, Providence, RI, 2012. x+282 pp. ISBN: 978-0-8218-7430-1