1 Stieltjes transform

From last time, we saw that it suffices to study the last entry of $\left(\frac{1}{\sqrt{n}}M_n - zI_n\right)^{-1}$.

Definition 1.1. *Let M be a $(p+q)\times(p+q)$ matrix, and write*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A is $p\times p$ and D is $q\times q$. Then the Schur complement of A is the $q\times q$ matrix $M/A := D - CA^{-1}B$.

If A is invertible, then

$$M^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B(M/A)^{-1}CA^{-1} & -A^{-1}B(M/A)^{-1} \\ -(M/A)^{-1}CA^{-1} & (M/A)^{-1} \end{pmatrix}.$$

Apply to our situation, if we write

$$M_n = \begin{pmatrix} M_{n-1} & X \\ X^* & \xi_{n,n} \end{pmatrix},$$

we see that

$$\left(\frac{1}{\sqrt{n}}M_n - zI_n\right)^{-1}_{n,n} = \left(\frac{1}{\sqrt{n}}\xi_{n,n} - z - \frac{1}{n}X^*\left(\frac{1}{\sqrt{n}}M_{n-1} - zI_{n-1}\right)^{-1}X\right)^{-1}.$$

As we have seen before, to prove the semicircle law, we may assume that the diagonal entries are 0. Therefore,

$$\mathbf{E}\left[\left(\frac{1}{\sqrt{n}}M_n - zI_n\right)^{-1}_{n,n}\right] = -\mathbf{E}\left(z + \frac{1}{n}X^*\left(\frac{1}{\sqrt{n}}M_{n-1} - zI_{n-1}\right)^{-1}X\right)^{-1}. \tag{1.1}$$

Write $R = \left(\frac{1}{\sqrt{n}}M_{n-1} - zI_{n-1}\right)^{-1}$. We would like to know what X^*RX is. First, observe that X and R are independent. Therefore, we may condition on R and study X^*RX assuming R is some deterministic matrix. But before that, we also observe that M_{n-1} is Hermitian, and hence all its eigenvalues are real. In particular, this implies the operator norm of R, $\|R\|$, is of order $O(1)$, since we assume that the imaginary part of z is positive. So it suffices to understand what X^*RX is when R is deterministic and $\|R\| = O(1)$.

Before study X^*RX, let's first study X^*AX, where A is a positive semidefinite matrix with $\|A\| = O(1)$. Also, we will further assume that the entries of M_n are uniformly bounded (which is fine by the reduction we saw before). In this case, the map $X \mapsto (X^*AX)^{1/2} = \|A^{1/2}X\|$ is Lipschitz. Therefore, we can apply Talagrand's inequality to see

$$\mathbf{P}\left(\left|(X^*AX)^{1/2} - \mathbf{M}(X^*AX)^{1/2}\right| \geq \lambda\right) \leq Ce^{-c\lambda^2}.$$

If A has k nonzero eigenvalues, then using Hoeffding's inequality one can show that $\|A^{1/2}X\| \geq \Omega(\sqrt{k})$ with high probability, and this also implies $\mathbf{M}(X^*AX)^{1/2} \geq \Omega(\sqrt{k})$. Moreover, observe that median satisfies $(\mathbf{M}(X^*AX)^{1/2})^2 = \mathbf{M}(X^*AX)$. Therefore, multiplying both sides in the probability by $\left|(X^*AX)^{1/2} + \mathbf{M}(X^*AX)^{1/2}\right|$, we see that

$$\mathbf{P}\left(|X^*AX - \mathbf{M}(X^*AX)| \geq \lambda\sqrt{k}\right) \leq Ce^{-c\lambda^2},$$

for some possibly different C and c. If A is Hermitian instead of positive definite, we may write $A = A_+ + A_-$, where A_+ has only nonnegative eigenvalues and A_- has only nonpositive eigenvalues, and applying triangle inequality we obtain

$$\mathbf{P}\left(|X^*AX - \mathbf{M}(X^*AX)| \geq \lambda\sqrt{n}\right) \leq Ce^{-c\lambda^2}$$

for some different C and c. Using the fact that any random variable Y with finite second moment satisfies $|\mathbf{M}Y - \mathbf{E}Y| = O(\mathrm{Var}(Y)^{1/2})$, we can replace median by the mean. If R is a general matrix with $\|R\| = O(1)$, we may write $R = A + B$, where A is Hermitian and B is skew-Hermitian, and similarly we will obtain

$$\mathbf{P}\left(|X^*RX - \mathbf{E}X^*RX| \geq \lambda\sqrt{n}\right) \leq Ce^{-c\lambda^2}.$$

Now, since $\xi_{i,j}$ has mean zero and variance 1, we have

$$\begin{aligned}
\mathbf{E}X^*RX &= \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} \mathbf{E}\overline{\xi_{i,n}} r_{i,j} \xi_{j,n} \\
&= \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} r_{i,j}\delta_{ij} \\
&= \mathrm{tr}(R).
\end{aligned}$$

Thus, we have

$$\mathbf{P}\left(|X^*RX - \mathrm{tr}(R)| \geq \lambda\sqrt{n}\right) \leq Ce^{-c\lambda^2}.$$

The above bound holds for deterministic matrix R with $\|R\| = O(1)$. If R is random, then

$$\mathbf{P}\left(|X^*RX - \mathrm{tr}(R)| \geq \lambda\sqrt{n}\right) = \mathbf{E}\left[\mathbf{P}\left(|X^*RX - \mathrm{tr}(R)| \geq \lambda\sqrt{n}\,|R\right)\right] \leq Ce^{-c\lambda^2}.$$

By some computations, we have

$$\mathrm{tr}(R) = \sqrt{n(n-1)}s_{n-1}\left(\frac{\sqrt{n}}{\sqrt{n-1}}z\right).$$

This expression is exactly the same as what we saw last time, and hence by what we showed last time, we have

$$\mathrm{tr}(R) = n(s_n(z) + o(1)).$$

Also, recall that almost surely, $s_n(z) - \mathbf{E}s_n(z) \to 0$ as $n \to \infty$, and hence

$$\mathrm{tr}(R) = n(\mathbf{E}s_n(z) + o(1)).$$

Finally, writing $E_n = \{|X^*RX - \mathrm{tr}(R)| \geq n^{1/3}\}$, and recalling the left hand side of (1.1) is $\mathbf{E}s_n(z)$, we obtain

$$\begin{aligned}
\mathbf{E}s_n(z) &= -\mathbf{E}\left(z + \frac{1}{n}X^*RX\right)^{-1} \\
&= -\mathbf{E}\left[\left(z + \frac{1}{n}X^*RX\right)^{-1}\mathbf{1}_{E_n}\right] + o(1) \\
&= -\mathbf{E}\left(z + \frac{1}{n}(n\mathbf{E}s_n(z) + o(1))\right)^{-1} + o(1) \\
&= -\frac{1}{z + \mathbf{E}s_n(z)} + o(1).
\end{aligned}$$

It is not difficult to show that $\mathbf{E}s_n$ is locally uniformly equicontinuous and locally uniformly bounded away from the real line. By the Arzelà-Ascoli theorem, $\mathbf{E}s_n$ converges locally uniformly to a limit s along a subsequence. So we have

$$s(z) = -\frac{1}{z + s(z)}.$$

Solving for $s(z)$, we have

$$s(z) = \frac{-z \pm \sqrt{z^2 - 4}}{2}.$$

Since the Stieltjes transform goes to 0 as $z \to \infty$, we conclude that

$$s(z) = \frac{-z + \sqrt{z^2 - 4}}{2}.$$

As there is only one possible subsequential limit of $\mathbf{E}s_n$, we conclude that $\mathbf{E}s_n$ converges locally uniformly to s, and thus $s_n(z)$ converges to $s(z)$ almost surely.

To finish the proof, it remains to find which distribution has the Stieltjes transform s, but this can be found by observing

$$\frac{s(\cdot + bi) - s(\cdot - bi)}{2\pi i} \Rightarrow \mu_{\mathrm{sc}}$$

as $b \downarrow 0$.

References

[1] Tao, Terence. Topics in random matrix theory. Graduate Studies in Mathematics, 132. American Mathematical Society, Providence, RI, 2012. x+282 pp. ISBN: 978-0-8218-7430-1

[2] Thompson, Brady. Talk on 4/13.