1 Tracy-Widom distribution

We saw that the largest eigenvalue (in magnitude) of a Hermitian Wigner matrix is around $2\sqrt{n}$. In fact, under some assumptions, we can even show that there is a central limit theorem. For simplicity we will consider only GUE.

Let $\mathrm{Ai}(x)$ denote the Airy function which is the unique solution of the Airy equation with prescribed asymptotics

$$\mathrm{Ai}''(x) = x\mathrm{Ai}(x), \qquad \mathrm{Ai}(x) \sum \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}x^{1/4}} \quad \text{as } x \to +\infty.$$

Let $u(x)$ be the unique monotone decreasing solution to the Painlevé II equation with asymptotics

$$u''(x) = 2(u(x))^3 + xu(x), \qquad u(x) \sim \mathrm{Ai}(x) \quad \text{as } x \to +\infty.$$

This u also satisfies

$$u(x) \sim -\sqrt{\frac{-x}{2}}(1 + O(x^{-2})) \quad \text{as } x \to -\infty.$$

Therefore, the function

$$F(x) := \exp\left(-\int_x^\infty (y - x)(u(y))^2 \, \mathrm{d}y\right)$$

defines a distribution function, called the Tracy-Widom distribution.

Theorem 1.1. *Let $\lambda_n(M)$ be the largest eigenvalue of an $n \times n$ GUE matrix. Then for all $x \in \mathbb{R}$,*

$$\lim_{n\to\infty} \mathbf{P}((\lambda_n(M) - 2\sqrt{n})n^{1/6} \le x) = F(x).$$

An interesting fact is that this distribution appears in many problems in probability, and they look unrelated to random matrices at first glance. Here we will introduce some of them.

2 Ulam's problem

Consider the symmetric group S_n and let $\pi \in S_n$. For $i_1 < i_2 < \cdots < i_k$, we say that $\pi(i_1), \ldots, \pi(i_k)$ is an increasing subsequence of π of length k if $\pi(x_1) < \cdots < \pi(i_k)$. Let $\ell_n(\pi)$ denote the maximal length of all increasing subsequence of π. An increasing subsequence of π of length $\ell_n(\pi)$ is called a longest increasing subsequence of π. For example, for the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 3 & 2 & 4 \end{pmatrix},$$

134 and 124 are longest increasing subsequences of π. We will pick a permutation uniformly at random from S_n, how does the corresponding random variable ℓ_n behave? Surprisingly, it turns out that it behaves like the largest eigenvalue of GUE.

Theorem 2.1. *For any $x \in \mathbb{R}$,*

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{\ell_n(\pi) - 2\sqrt{n}}{n^{1/6}} \le x\right) = F(x).$$

3 Last-passage percolation

Consider the integer lattice $\mathbb{Z}^2$. Let $(t_x)_{x\in\mathbb{Z}^2}$ be a family of i.i.d. random variables. An oriented path γ with vertices $x_0, \dots, x_n$ has the property that all coordinates of x_i are no larger than those of x_{i+1}. For an oriented path γ, we define the passage time $T(\gamma) = \sum_{x\in\gamma\setminus\{x_0\}} t_x$. Finally, we define the last-passage time between x and y by

$$T(x,y) = \max\{T(\gamma) : \gamma \text{ is an oriented path from } x \text{ to } y\}.$$

Of course, this only makes sense when there is an oriented path from x to y. Again, this model is related to Tracy-Widom distribution.

Theorem 3.1. *Suppose that t_x is either exponentially distributed with parameter 1 or geometrically distributed with parameter p. Then for all $x \in \mathbb{R}$,*

$$\lim_{m,n\to\infty, m/n\to\alpha\in(0,1)} \mathbf{P}\left(\frac{T((0,0),(m,n)) - g(m,n)}{\sigma(m,n)} \le x\right) = F(x),$$

where

$$g(m,n) = \begin{cases} (\sqrt{m}+\sqrt{n})^2 & \text{if } t_x \text{ is exponential,} \\ \dfrac{2\sqrt{mnp}+(m+n)p}{1-p} & \text{if } t_x \text{ is geometric,} \end{cases}$$

and

$$\sigma(m,n) = \begin{cases} \dfrac{(\sqrt{m}+\sqrt{n})^{4/3}}{(mn)^{1/6}} & \text{if } t_x \text{ is exponential,} \\ \dfrac{p^{1/6}(\sqrt{m}+\sqrt{np})^{2/3}(\sqrt{n}+\sqrt{mp})^{2/3}}{(1-p)(mn)^{1/6}} & \text{if } t_x \text{ is geometric.} \end{cases}$$

3.1 TASEP

There are some models that are equivalent to the last-passage percolation. We will describe one of them, TASEP.

The total asymmetric simple exclusion process (TASEP) is an important stochastic interacting particle systems. At any time, each site $j \in \mathbb{Z}$ is either occupied by a particle or is empty. Let $\eta(t) = \{\eta_j(t)\}_{j\in\mathbb{Z}}$ be defined by $\eta_j(t) = 1$ if site j has a particle at time t and 0 otherwise. Particles can jump to the site on the right if the site on the right is empty. When the right site is empty, a jump is performed after an exponential waiting time with mean 1. All jumps are independent of each other.

Suppose that at each site $j \in \mathbb{Z}$ with $j \le 0$, there sits a particle at time 0. The particle at site 0 is allowed to move without any restriction, but other particles may sometimes be blocked.

TASEP is related to the last-passage percolation for the following reason. Consider last-passage percolation with exponential weights. The procession of the first particle in TASEP is the same as last-passage time from $(0, 0)$ along the x-direction. For the second level (that is, $y = 1$), the number of sites reached in last-passage percolation corresponds to the number of steps the second particle moved in TASEP. In general, the n-th step of the k-th particle in TASEP corresponds to the site $(n - 1, k)$ being reached from $(0, 0)$. This shows why these two models are equivalent. In fact, the formula of g (the exponential case) in Theorem 3.1 was actually obtained using TASEP.

References

[1] Baik, Jinho; Deift, Percy; Suidan, Toufic. Combinatorics and random matrix theory. Graduate Studies in Mathematics, 172. American Mathematical Society, Providence, RI, 2016. xi+461 pp. ISBN: 978-0-8218-4841-8