# The travel time to infinity in percolation

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Based on joint work with M. Damron, J. Hanson and X. Wang

## Introduction

Consider $(\mathbb{Z}^2, \mathcal{E}^2)$. Let $(t_e)_{e \in \mathcal{E}^2}$ be i.i.d. nonnegative edge weights. For a lattice path $\gamma$, define $T(\gamma) = \sum_{e \in \gamma} t_e$.

**Questions**:

1. $\exists$ an infinite self-avoiding path $\gamma$ starting at 0 such that $T(\gamma) < \infty$?
2. For which distribution $F$ of $t_e$ does such a path exist, where $F(t) = \mathbf{P}(t_e \le t)$?

A generalization of Bernoulli bond percolation.

- Easy case: $F(0) > p_c = \frac{1}{2}$. In this case,

  $$\mathbf{P}(\exists \text{ infinite path of } e \text{ with } t_e = 0) = 1.$$

  Connect such an infinite path to 0. Answer to 1. is yes.
- Similarly, if $F(0) < p_c$, then answer is no.
- **Difficult case**: $F(0) = p_c$.

## Reformulation

Consider $\partial B(n) := \{x \in \mathbb{Z}^2 : \|x\|_\infty = n\}$. Define

$$T(0, \partial B(n)) = \inf_{\gamma : 0 \to \partial B(n)} T(\gamma).$$

By monotonicity, $\rho := \lim_{n \to \infty} T(0, \partial B(n))$ exists.

Fact: the answer to 1. is yes if and only if $\rho < \infty$.

**Question**: When is $\rho < \infty$?

## Previous work

From now on, always assume $F(0) = p_c$.

- (Chayes-Chayes-Durrett, '86) $t_e$ Bernoulli $\Rightarrow \mathbf{E}T(0, \partial B(n)) \asymp \log n$.
- (Chayes, '91) $\forall \delta > 0$, $T(0, \partial B(n))/n^\delta \to 0$. (Also true in higher dimensions.)
- (Kesten-Zhang, '97) Showed a Gaussian CLT for $T(0, \partial B(n))$ when $t_e$ is "almost" Bernoulli.

## Previous work (cont.)

(Zhang, '99) It is possible to have $\rho < \infty$ or $\rho = \infty$. Specifically, for $a > 0$, define

$$F_a(t) = \begin{cases} 1 & \text{if } t^a > 1 - p_c, \\ t^a + p_c & \text{if } 0 \le t^a \le 1 - p_c, \\ 0 & \text{otherwise,} \end{cases}$$

and for $b > 0$, define

$$G_b(t) = \begin{cases} 1 & \text{if } e^{-\frac{1}{t^b}} > 1 - p_c, \\ e^{-\frac{1}{t^b}} + p_c & \text{if } 0 \le e^{-\frac{1}{t^b}} \le 1 - p_c, \\ 0 & \text{otherwise.} \end{cases}$$

- For $a$ small, if $t_e \sim F_a$, then $\rho < \infty$.
- If $b > 1$ and $t_e \sim G_b$, then $\rho = \infty$.

Intuition: When $a$ is small, w.h.p., $t_e \approx 0$. Can construct an infinite path such that most edges take extremely low weights.

When $a$ is large, $F_a \approx G_b$. Zhang conjectured the following:

**Conjecture** (Zhang). *For sufficiently large $a > 0$, if $t_e \sim F_a$, then $\rho = \infty$.*

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(Yao, '14, '18) Consider the triangular lattice $\mathbb{T}$ and put the random weights on the vertices instead of the edges.

**Theorem** (Yao). *<u>On $\mathbb{T}$</u>, if $t_v$ is Bernoulli with $F(0) = p_c$, then*

*(a)*

$$\frac{T(0, \partial B(n))}{\log n} \to \frac{1}{2\sqrt{3}\pi} \quad \text{a.s.,}$$

*(b)*

$$\frac{\mathrm{Var}(T(0, \partial B(n)))}{\log n} \to \frac{2}{3\sqrt{3}\pi} - \frac{1}{2\pi^2}.$$

- (a) can be seen as a strong law of large numbers.
- The proof uses the conformal loop ensemble of Camia and Newman.

## Main result 1: asymptotics

**Theorem 1** (Damron-L.-Wang, '17). *For $t \in (0, 1)$, define $F^{-1}(t) = \inf\{x : F(x) \ge t\}$. Suppose that $F(0) = p_c$ and write $X_n = T(0, \partial B(n))$. Then the following holds:*

- *$\rho < \infty$ if and only if $\sum_n F^{-1}(p_c + 2^{-n}) < \infty$.*
- *$\mathbf{E}X_1^{1+\varepsilon} < \infty \Rightarrow \mathbf{E}X_n \asymp \sum_{k=2}^{\log n} F^{-1}(p_c + 2^{-k})$.*
- *If $\mathbf{E}X_1^{2+\varepsilon} < \infty$, then*

  $$\mathrm{Var}(X_n) \asymp \sum_{k=2}^{\log n} (F^{-1}(p_c + 2^{-k}))^2.$$

- *If $\mathbf{E}X_1^{2+\varepsilon} < \infty$, $\mathbf{E}X_n, \mathrm{Var}(X_n) \to \infty$, then*

  $$(X_n - \mathbf{E}X_n)/\sqrt{\mathrm{Var}(X_n)} \Rightarrow N(0, 1).$$

- *If $\mathbf{E}X_1^{2+\varepsilon} < \infty$, $\mathbf{E}X_n \to \infty$ but the variance is bounded, then $X_n - \mathbf{E}X_n \to Z$ for some r.v. $Z$.*

**Remark.** *Theorem 1 holds on any two dimensional lattice.*

Applications:

- $t_e$ Bernoulli: $F^{-1}(p_c + 2^{-n}) = 1$, so $\rho = \infty$.
- If $F$ has a positive right derivative at 0 and $t_e \sim F$, then $\sum_n F^{-1}(p_c + 2^{-n}) < \infty$, and hence $\rho < \infty$.
- $t_e \sim F_a$: $F^{-1}(p_c + 2^{-n}) = 2^{-n/a}$, which is summable for all $a$. Therefore, $\rho < \infty$ for all $a$, and hence <u>**Zhang's conjecture is false**</u>.
- $t_e \sim G_b$: $F^{-1}(p_c + 2^{-n}) \approx n^{-1/b}$. Thus $\rho = \infty$ if and only if $b \ge 1$.

## Main result 2: universality

Further universality results on the triangular lattice that improve Yao's results: define $I = \inf\{x > 0 : F(x) > p_c\}$, the infimum of the support of the law of $t_v$ excluding 0.

**Theorem 2** (Damron-Hanson-L., '19). *<u>On $\mathbb{T}$</u>, when $F(0) = p_c$, we have*

$$\frac{T(0, \partial B(n))}{\log n} \to \frac{I}{2\sqrt{3}\pi} \quad \text{almost surely.}$$

*If we further assume $\mathbf{E}T(0, \partial B(1))^2 < \infty$, then*

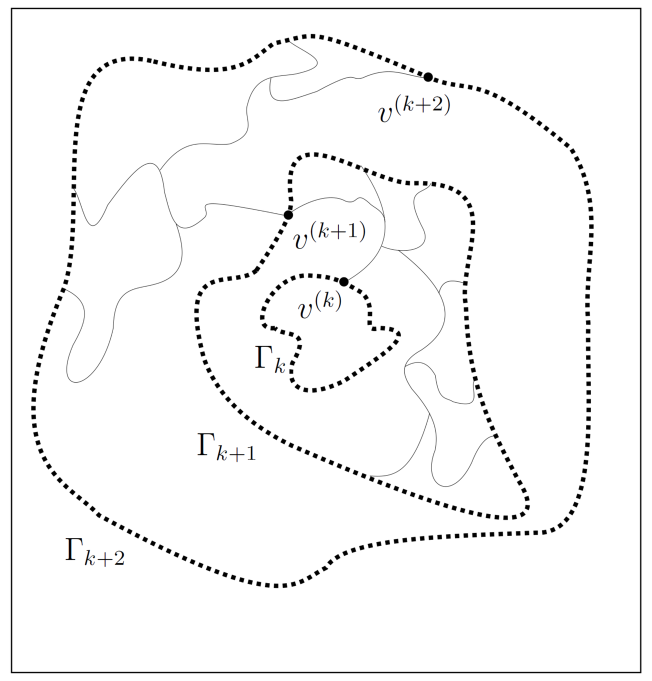
$$\frac{\mathrm{Var}(T(0, \partial B(n)))}{\log n} \to I^2 \left( \frac{2}{3\sqrt{3}\pi} - \frac{1}{2\pi^2} \right).$$

**Remark.**

1. *If one can prove the limit theorems for Bernoulli weights on $\mathbb{Z}^2$, then the limit theorems hold for general weights on $\mathbb{Z}^2$.*
2. *Does*

   $$\lim_{n \to \infty} \frac{T(0, \partial B(n))}{\sum_{k=2}^{\log n} F^{-1}(p_c + 2^{-k})} \quad \text{exist}$$

   *when $I = 0$? ($\log n$ is not the correct order.)*




*Idea of the proof.* For simplicity, assume that $I = 1$. Say a vertex $v$ is open if $t_v = 0$, closed if $t_v \ge 1$.

- Using tools of Kesten-Sidoravicius-Zhang: can construct closed circuits $(\Gamma_k)$ that surround 0; between $\Gamma_k$ and $\Gamma_{k+1}$, w.h.p. $\exists$ large open cluster that consists of a lot of branchings.
- Can construct a path using these open clusters. The path gains weight $\ge 1$ when it passes $\Gamma_k$, but can be made $\approx 1$.
- The path constructed is a Bernoulli geodesic (under a suitable coupling). Original passage time $\approx$ Bernoulli passage time.